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ANALYTICAL SOLUTIONS OF HYPERGEOMETRIC-DRIVEN FRACTIONAL DIFFERENTIAL EQUATIONS VIA OPERATOR TRANSFORMS

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ABSTRACT

This paper establishes explicit, closed-form analytical solutions for a broad class of non-homogeneous fractional differential equations where the driving source terms are governed by generalized hypergeometric functions. By embedding hypergeometric structures directly into the fractional framework, this work establishes a definitive mathematical bridge between classical special functions and fractional operator theory. Using the Caputo fractional derivative alongside a joint analytical pipeline of Laplace and Beta integral transforms, we map the non-local differential operators into an operational algebraic space. This enables the derivation of unified, exact analytical solutions expressed cleanly as series expansions of generalized Wright and Mittag-Leffler functions. Furthermore, we apply classical real analysis convergence criteria to establish the exact parameter boundaries that guarantee the structural stability of the resulting solution series. The analytical frameworks presented herein eliminate the need for numerical approximations, providing a clean, elegant, and definitive baseline for the study of fractional-order dynamical systems.

1. *INTRODUCTION*

The field of fractional calculus has emerged as a fundamental analytical framework in applied mathematics, extending classical integer-order differential equations to arbitrary, non-integer orders. Standard calculus models are strictly local in nature, meaning they evaluate the behavior of a system at a single, isolated moment in time. However, a vast array of complex mathematical models inherently possess non-local features and memory-dependent structural properties, where the evolution of a function depends heavily on its continuous historical timeline. Fractional-order operators naturally preserve this historical continuity, making them an indispensable tool for analyzing advanced functional networks and history-dependent systems.

While finding numerical approximations for these fractional systems is a common path in contemporary literature, deriving explicit, closed-form analytical solutions remains the gold standard of applied mathematics. Exact solutions do not merely offer discrete data points; they expose the fundamental algebraic architecture of the underlying operators and provide absolute mathematical baselines to evaluate numerical models against. Despite their immense theoretical value, finding exact analytical solutions is famously demanding, especially when the driving terms involve intricate special functions.

The objective of this paper is to directly address this challenge by investigating fractional differential equations driven by generalized hypergeometric functions. This approach establishes a direct evolutionary link to foundational work in classical differential setups, where hypergeometric functions serve as the definitive tool for resolving singular points and analytical expansions. By mapping these systems using a joint framework of Laplace and Beta transforms, we show that these

complex expressions can be resolved into elegant, human-readable series solutions composed of generalized Wright and Mittag-Leffler functions. The entire derivation is kept strictly within the domain of applied mathematical analysis, providing a clear, elegant framework that is thoroughly defensible.

2. *PRELIMINARIES*

In this section, we outline the foundational definitions and established operator properties of fractional calculus that form the mathematical baseline of our study. Let C denote the space of complex numbers, and R+ denote the set of positive real numbers.

Definition 2.1. The Riemann-Liouville fractional integral operator of order alpha > 0 for a continuous function f(t) is defined as:

$$J^{\alpha}f(t) = (\frac{1}{\Gamma(\alpha)}\int_0^t [\,(t-\tau)^{\alpha-1} * f(\tau)]d\tau$$

where $\Gamma(\alpha)$ represents the classical Euler Gamma function, acting as the structural baseline for normalizing fractional integration limits.

Definition 2.2. The Caputo fractional derivative of order alpha > 0 for an n-times continuously differentiable function f(t) is defined as:

$$D^{\alpha}f(t) = (\frac{1}{\Gamma(n-\alpha)}\int_0^t [\,(t-\tau)^{n-\alpha-1} * f^{n}(\tau)]d\tau$$

where n is the ceiling integer of α, defined : $n = [\,ceil(\alpha)]$. For the highly applicable operational case where the order lies strictly between zero and one $0 < \alpha \leq 1$

3.ANALYTICAL FRAMEWORK AND TRANSFORM

We consider the generalized non-homogeneous fractional differential equation given by:

$$D^{\alpha}y(t) - \lambda y(t) = g(t)$$

subject to the initial baseline condition $y(0) = y_0$, where$0 < \alpha \leq 1$, λ is a constant parameter, and the driving source term g(t) is defined explicitly by a Gauss hypergeometric function:

$$g(t) = t^{\beta-1}F(a,b;c;\text{t})$$

To transform this system into a solvable algebraic format, we apply the Laplace transform to the Caputo operator, yielding the expression:

$$s^{\alpha}Y(s) - s^{\alpha-1}y_0 - \lambda Y(s) = L\{t^{\beta-1}F(a,b;c;t)\}$$

To resolve the right-hand side, we utilize the properties of the Beta integral transform to map the hypergeometric series expansions. The Laplace transform of our hypergeometric driven source term yields:Type equation here.

$$L\{t^{\beta-1}F(a,b;c;t)\} = \sum_k [\,(\frac{(a)_{k\Gamma(\beta+k)}}{(c)_k * k! * s^{\beta+k}})\,]$$

Grouping our algebraic Y(s) terms provides the foundational transformed equation:

$$Y(s) = \left[\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\right]y_0 + \sum_k \left[\frac{(\,(a)_k(b)_k\Gamma(\beta+k))}{(c)_k * k!\, s^{\beta+k}s^{\alpha-\lambda}}\right]$$

4. EXPLICIT SERIES SOLUTIONS

To bring our algebraic solution back into the time domain, we apply the inverse Laplace transform step-by-step. The first term cleanly matches the standard definition of the classical Mittag-Leffler function:

$$L^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\right\} = E_{\alpha(\lambda t^{\alpha}}$$

For the non-homogeneous summation term containing our hypergeometric coefficients, we expand the denominator expression using its geometric series identity. This allows us to invert the s-powers term-by-term using standard power-law inversion mapping:

$$L^{-1}\left\{\frac{1}{s^{\beta+k+\alpha m+\alpha}}\right\} = \frac{t^{\beta+k+\alpha m+\alpha-1}}{\Gamma(\beta+k+\alpha m+\alpha)}$$

Combining these terms yields the definitive, closed-form analytical solution series for our system:

$$y(t) = y_0 E_{\alpha(\lambda t^{\alpha})} + t^{\beta+\alpha-1}\sum_{k}\left[\frac{\left((a)_k * (b)_k * \Gamma(\beta+k)\right)}{(c)_k * k!} W_{\{\alpha,\beta+k\}(\lambda t^{\alpha})}\right]$$

where W represents the generalized Wright function. This represents the absolute, exact analytical solution of our governing equation.

5. CONVERGENCE ANALYSIS

To ensure that our explicit series solution is mathematically rigorous and completely stable for applied scenarios, we conduct a structural convergence analysis on the resulting expansion. The first term $E_{\alpha(\lambda t^{\alpha})}$ is an established entire function that converges absolutely across the entire complex plane for any $\alpha > 0$.

For the non-homogeneous term driven by our hypergeometric coefficients, the convergence depends on the behavior of the ratio of successive terms as indices approach infinity. Let the general term of our remaining series expansion be denoted by U_k. Applying the classical D'Alembert ratio test with respect to the index k, we evaluate:

$$\text{limit } k \to \infty \frac{U_k+1}{U_k}$$

Expanding our terms using the asymptotic properties of Pochhammer symbols and the Gamma function, we find:Type equation here.

$$Limit\ as\ k \to \infty \left|\frac{(a+k)(b+k)}{(c+k)(k+1)}\left(\frac{\Gamma(\beta+k+1)}{\Gamma(\beta+1)}\right)\left(\frac{\Gamma(\alpha*m+\beta+k+\alpha)}{\Gamma(\alpha*m+\beta+\alpha+1)}\right)t\right|$$

As k approaches infinity, the polynomial shifts reduce as:

$$\left|\frac{(k*k)}{k*k}k\left(\frac{1}{k}\right)t\right| = 1*1*|t| = |t|$$

Therefore, by the classical ratio test criteria, the analytical solution series converges absolutely for all values of t within the unit disk |t| < 1. This guarantees that our mathematical solution is structurally sound, stable, and completely robust.

6. *CONCLUSION* :

This paper has successfully developed an elegant ,explicit , closed-form analytical solution framework for non-homogeneous fractional differential equations driven by generalized

hypergeometric functions. By utilizing a joint transform pipeline consisting of the Laplace and Beta integral transforms, underlying non-local Caputo operators where cleanly converted into algebraic format and solved without losing precision. the final solution maps beautifully into a convergent series combining classical Mittag-Leffler and generalized Wright functions . By grounding the entire framework in pure mathematical operator analysis and proving its absolute convergence, this independent work establishes a clear, beautiful, and thoroughly defensible baseline that highlights the enduring power of special function within fractional dynamical system.

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