International Research Journal of Management Science & Technology

ISSN 2250 – 1959(0nline)
2348 – 9367 (Print)

An Internationally Indexed Peer Reviewed & Refereed Journal

www.IRJMST.com
www.isarasolutions.com

Published by iSaRa Solutions

"Second Order Quadratic Inclusions in Banach Algebra"

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For the Degree of Doctor of Philosophy in Mathematics

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1. Abstract:

The study of existence the solution of second order quadratic functional nonlinear inclusion in Banach algebras with existence the extremal solution of nonlinear quadratic functional differential inclusions. We also used some fixed point theorem. we discuss on basic concept needed for multivalued analysis and fixed point theory.Some examples are included in this paper.

(AMS) subject classification: 26A33, 47H10, 34K09, 34A12, 26A33, 37C25, 54H25

2. Introduction

In the mathematical modelling, we have integral, differential inclusion phenomenon. Some included problem in economics, optimal control theory, game theory. Most of the natural and physical phenomena in the universe involve the decay or growth that is the change of state with respect to time period so such dynamical system in the universe governed by the nonlinear differential and integral inclusion.The useful application of integral inclusion arise in the system are governed by relay rheology, porous media, viscoelasticity, electrodynamics etc. Many mathematicians have contributed interest ing results mostly in the area of integral and differential inclusions. In the fractional calculus, we use nonlocal variables in fractional order inclusion has helped to improve the mathematical modelling of many real world problems. The study of all above mentioned works and mainly fixed point theorem or iterative methods. In this fixed-point method, which is powerful and effective tool. Some fixed-point theorems of single valued mapping in Banach spaces. The hybrid fixed point theorems for multivalued mappings are applicable to existence results under mixed Lipschitz, contraction and caratheodory condition. In this paper we shall converted differential inclusion with initial condition into fractional order quadratic functional nonlinear integral inclusions in Banach algebra the results have been obtained by applying hybrid fixed point theorem due to Dhages for two operators and three operators in Banach algebra and study of the extremal solution under certain monotonicity condition.

statement of the problem:

Consider fractional order hybrid quadratic functional differential inclusion.

$$D^{\alpha}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in f\left(t,x(\varphi_1(t))\right) \qquad t \in R^+ \quad (1)$$

Multivalued nonlocal such that the integral condition for first order x(0) = 0. Dα is Riemann derivative of α ∈ (0,1) ,f : R+ × R → P(R), g : R+ × R → R0 ϕ1,ϕ2 : R+ →R+ are two set valued mapping where P(R) denote family of nonempty subset of R.

3. PRELIMINARIES

In this section we collect notation, definitions, some remarks, hypothesis, and preliminary tool which will be needed for above problem.

Fractional calculus[1,15,23]:

Definition 3.1(23):

The Riemann- Liouville fractional integral of order ζ > 0 of a continuous function $f \in L_1[0,T]$ is defined by

$$I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} f(s) ds$$

Provided the right-hand side is point-wise defined on R^+

Definition 3.2(23):

The Riemann-Liouville fractional derivative of order $\xi > 0$ of the function $g \in L^1[0,T]$ defined by,

$$D^{\xi} g(t) = \frac{1}{\Gamma(n-\xi)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\xi-1} g(s) ds$$

$$n-1 < \xi < n$$

Where Γ denotes the Euler gamma function. The Reimann-Liouville fractional derivative operator of order ξ defined by $D^{\xi} = \frac{d^{\xi}}{dt^{\xi}} = \frac{d}{dt} \circ I^{1-\xi}$

Definition 3.3[20]:

Let X be a Banach space. A mapping $T: X \longrightarrow X$ is called **Lipschitz** if there exists a constant $\alpha > 0$ such that, $\|Tx - Ty\| \leq \alpha \|x - y\|$for all $x, y \in X$. If $\alpha < 1$, then T is called contraction on X with the contraction constant α.

Definition 3.4 [21]:

Let X be a Banach space. An operator $T: X \longrightarrow X$ is called **compact** if for any bounded subset S of X, $T(S)$is relatively compact subset of X. If T is continuous and compact, then it is called completely continuous on X.

Definition 3.5[22,25]: Let (X,d) be the metric space and $a \in X$ and for some real number $r > 0$ then the set $\|B_r(a)\| = sup\{x \in X: d(x,a) \leq r\}$ is called closed ball centered at a with radius r.

Definition 3.6[23,24]: The solution of inclusion is said to be locally attractive if there exists a closed ball $B_r(x_0)$ in the space $BC(\mathbb{R}^+, \mathbb{R})$ for some $x_0 \in BC(\mathbb{R}^+, \mathbb{R})$ and for real number $r > 0$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of inclusion belonging to $B_r(x_0) \cap C$ we have that $\lim_{t\to\infty} \big(x(t) - y(t)\big) = 0$.

Theorem 3.1 [26]: (Arzela-Ascoli Theorem) A metric space X is compact if and only if every sequence in X has a convergent subsequence.

Theorem 3.2[26]:

If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(R^+, R)$, then it has a convergent subsequence.

Multivalued Analysis (see [27,28,29,30,31])

Let X denote an ordered metric space with a metric d and an ordered relation $\leq$. Then X is an ordered topological space, where the topology on X is induced by the metric d on it. Let $\mathcal{P}(X)$ and $\mathcal{P}_p(X)$ denote, respectively, the class of all subsets and the class of all nonempty subsets of X with the property p. Thus, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cp}(X)$ and $\mathcal{P}_{cv}(X)$ denote respectively, the classes of all closed, bounded, compact and convex subsets of X.

Definition 3.7 [19]:

A mapping $Q: X \longrightarrow \mathcal{P}_p(X)$ is called a multivalued mapping or a multivalued operator on X and a point $u \in X$ is called a fixed point of Q if $u \in Qu$.

Let $(X, \|\cdot\|)$ be a norm space, $\mathcal{P}(X)$ denote the set of all non-empty subsets of X.

Let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X): Y \text{ is closed}\}$

$\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X): Y \text{ is bounded}\}$,

$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X): Y \text{ is compact}\}$,

$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X): Y \text{ is convex}\}$

$\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X): Y \text{ is compact and convex}\}$.

Let$Q: X \longrightarrow \mathcal{P}(X)$ be a multivalued map.

Definition 3.8 [30]:Q is convex valued if $Q(X)$ is convex for all $x \in X$.

Definition 3.9 [31]:Q is bounded on bounded set if $Q(Y) = \cup_{x\in Y} Q(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$, that is $sup_{x\in Y}\{sup\{|y|: y \in Q(x)\}\} < \infty$.

Definition 3.10 [31]:Q is upper semi-continuous (u. s. c.) on X if for each $x_0 \in X$, the set $Q(x_0)$ is a closed subset of X, and if for each open set N of X containing $Q(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $Q(\mathcal{N}_0) \subset \mathcal{N}$.

Definition 3.11 [31]:Q is compact if $\overline{Q(X)}$ is a compact subset of X.

Definition 3.12 [31]:Q is totally bounded if for any bounded subset S of X, $Q(S) = \cup_{x\in S} Qx$ is totally bounded subset of X.

We know that every compact multivalued operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of X.

Definition 3.13 [30]:Q is completely continuous if it is upper semicontinuous and compact multivalued operator on X.

Definition 3.14 [31]:Q is completely continuous if $Q(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$

Definition 3.15 [31]:

If the multi-valued map Q is completely continuous with nonempty compact values, then, Q is u.s.c. if and only if Q has a closed graph, that is, $x_n \to x_*$, $y_n \to y_*$, $y_n \in Q(x_n)$ implies that $y_* \in Q(x_*)$.

Definition 3.16 [31]: The multi-valued map Q has fixed point if there is $x \in X$ such that $x \in Q(x)$. The fixed-point set of Q will be denoted by $FixQ$.

Definition 3.17 [31]: The graph of a multivalued map $G: X \longrightarrow \mathcal{P}(Y)$ is the set $(G) = \{(x, y) \in X \times Y, y \in G(x)\}$.

Some Lemma's and theorems are used in sequel.

Lemma 3.1 [30]:

If $G: X \longrightarrow \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$ that is, for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \longrightarrow \infty$, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph then it is upper semi-continuous.

Lemma 3.2 [21]: Let X be a Banach space. Let $F: \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}_{cp,cv}(X)$ be an L^1- Caratheodory multivalued map and let Θ be a linear continuous mapping from $L^1(\mathbb{R}^+, X)$ to $C(\mathbb{R}^+, X)$. Then, the operator $\Theta \circ S_F: C(\mathbb{R}^+, X) \longrightarrow \mathcal{P}_{cp,cv}\big(C(\mathbb{R}^+, X)\big)$, $x \longmapsto (\Theta \circ S_F)(x) = \Theta\big(S_{F,x}\big)$ is a closed graph operator in $C(\mathbb{R}^+, X) \times C(\mathbb{R}^+, X)$.

Theorem 3.3 [18]:

Let X be a Banach algebra and let $\mathcal{A}, \mathcal{C}: X \to X$ be a single valued and $\mathcal{B}: X \to \mathcal{P}_{cp,cv}(X)$ be a multi-valued operatorsatisfying,

a) $\mathcal{A}$ and $\mathcal{C}$ are single valued Lipschitz with Lipschitz constants q_1 and q_2 respectively.
b) $\mathcal{B}$ is compact and upper semi-continuous.
c) $Mq_1 + q_2 < 1$, where $M = \|U\mathcal{B}(X)\|$

Then either

I. The operator inclusion $x \in \mathcal{A}x\mathcal{B}x + \mathcal{C}x$, has a solution, or
II. The set $\varepsilon = \{u \in X, \mu u \in \mathcal{A}u\mathcal{B}u \,, \mu > 1\}$ is unbounded.

4 Existence theory

Let $\mathbb{R}$ denote the real line. Given $\mathbb{R}^+$ is nonnegative real number in $\mathbb{R}$. We have the solution of inclusions in the space $BC(\mathbb{R}^+, \mathbb{R})$ of bounded and continuous real valued functions defined on $\mathbb{R}^+$. Define a norm $\|.\|$ and a multiplication "." in $BC(\mathbb{R}^+, \mathbb{R})$ by

$$\|x\| = sup_{t\in\mathbb{R}^+}|x(t)|$$

and $\quad (x.y)(t) = (xy)(t) = x(t)y(t)\,, t \in \mathbb{R}^+$ for all $x, y \in BC(\mathbb{R}^+, \mathbb{R})$.

Then $BC(\mathbb{R}^+, \mathbb{R})$ is Banach algebra w. r. t. to the above norm and multiplication in it.

By $L^1(\mathbb{R}^+, \mathbb{R})$ we denote the space of Lebesgue-integrable function on $\mathbb{R}^+$ with the norm $\|.\|_{L^1}$ defined by $\|x\|_{L^1} = \int_0^t |x(t)|dt$

Definition 4.1 [20]:A multivalued map $F: \mathbb{R}^+ \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Caratheodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in \mathbb{R}^+$;

Further a Caratheodory function F is called L^1-Caratheodory if,

(iii) For each real number $r > 0$ there exists a function $h_r \in L^1(\mathbb{R}^+, \mathbb{R})$ such that $\|F(t,x)\| = sup\{|v|: v \in F(t,x)\} \leq h_r(t)$ a. e. $t \in \mathbb{R}^+$ for all $x \in \mathbb{R}$ with $\|x\| \leq r$

Finally, a Caratheodory function F is called L^1_X -Caratheodory if,

(iv) There exists a function $h \in L^1(\mathbb{R}^+, \mathbb{R})$ such that $\|F(t,x)\| = sup\{|v|: v \in F(t,x)\} \leq h(t)$ a. e. $t \in \mathbb{R}^+$ for all $x \in \mathbb{R}$.

Definition 4.2 [30,31]:

For each $\in \mathbb{R}$, define the set of selections of F by

$$S_{F,x} \coloneqq \{v \in L^1(\mathbb{R}^+, \mathbb{R}): v(t) \in F\big(t, x(t)\big)\, for\ a.e. t \in \mathbb{R}^+\}$$

Lemma 4.1:-

If $f \in L^1(R^+, R)$ if and only if x is a solution of the second order quadratic functional nonlinear differential inclusion.

$$\frac{d^2}{dt^2}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in f\left(t,x(\varphi_1(t))\right) \qquad t \in R^+$$

$$x(t) \in g\left(t,x(\varphi_2(t))\right)\int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right) ds\ \ ds \qquad (2)$$

Proof: consider the given differential inclusion is

$$\frac{d^2}{dt^2}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in f\left(t,x(\varphi_1(t))\right) \qquad t \in R^+$$

Taking integration from 0 to t we have ,

$$\int_0^t \frac{d^2}{dt^2}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in \int_0^t f\left(s,x(\varphi_1(s))\right)\, ds$$

$$\frac{d}{dt}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in \int_0^t f\left(s,x(\varphi_1(s))\right)\, ds$$

Again integrate from 0 to t we have,

$$\int_0^t \frac{d}{dt}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in \int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

$$\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)} \in \int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

$$x(t)-x(0) \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

$$x(t)-x(0) \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t (t-s)\ f\left(s,x(\varphi_1(s))\right)\, ds$$

With the condition $x(0) = 0$ we have ,

$$x(t) \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t (t-s)\ f\left(s,x(\varphi_1(s))\right)\, ds$$

Conversely suppose that,

$$x(t) \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t (t-s)\ f\left(s,x(\varphi_1(s))\right)\, ds$$

$$x(t)-0 \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

With initial condition $x(0) = 0$ we have,

$$x(t)-x(0) \in\ g\left(t,x(\varphi_2(t))\right)\int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

$$\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)} \in \int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

Now differentiate with respect to t we have,

$$\frac{d}{dt}\left(\frac{x(t)-x(0)}{g\left(t,x(\varphi_2(t))\right)}\right) \in \frac{d}{dt}\int_0^t\int_0^t f\left(s,x(\varphi_1(s))\right)\, ds\, ds$$

$$\frac{d}{dt}\left(\frac{x(t)-x(0)}{g\left(t,x\big(\varphi_2(t)\big)\right)}\right) \in \int_0^t f\left(s,x\big(\varphi_1(s)\big)\right)\,ds$$

Again differentiate with respect to t we have,

$$\frac{d^2}{dt^2}\left(\frac{x(t)-x(0)}{g\left(t,x\big(\varphi_2(t)\big)\right)}\right) \in \frac{d}{dt}\int_0^t f\left(s,x\big(\varphi_1(s)\big)\right)\,ds$$

$$\frac{d^2}{dt^2}\left(\frac{x(t)-x(0)}{g\left(t,x\big(\varphi_2(t)\big)\right)}\right) \in f\left(t,x\big(\varphi_1(t)\big)\right)\,dt \qquad ,t \in R^+$$

With condition $x(0) = 0$.

5. Main results

5.1 Existence and locally attractive solution of Second order quadratic Functional Nonlinear Integral Inclusion

Definition 5.1.1: A function $x \in C(\mathbb{R}^+,\mathbb{R})$ is said to be a solution of second order quadratic functional nonlinear integral inclusion (4.1) if it satisfies,

$$x(t) = g\left(t,x\big(\varphi_2(t)\big)\right)\left(\int_0^t \int_0^t f\left(s,x\big(\varphi_1(s)\big)\right)ds\ ds\right)$$

for some $f \in L^1(\mathbb{R}^+,\mathbb{R})$ with $f \in F\left(t,x\big(\varphi_1(t)\big)\right)$ a. e. $t \in \mathbb{R}^+$.

Remark 4.1:

$$\int_0^t f(t)dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds$$

Using above equation becomes

$$x(t) = g\left(t,x\big(\varphi_2(t)\big)\right)\left(\int_0^t (t-s) f\left(s,x\big(\varphi_1(s)\big)\right)ds\right)$$

We require the following hypothesis for existence of solution of second order quadratic functional nonlinear integral inclusion (2).

(H5) The function $g: \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ is continuous and there exists a bounded function φ with bound $\|\varphi\|$ such that $\varphi(t) > 0$ a. e. $t \in \mathbb{R}^+$ and

$|g(t,x) - g(t,y)| \le \varphi(t)\left|x\big(\varphi_2(t)\big) - y\big(\varphi_2(t)\big)\right|$ a. e. $t \in \mathbb{R}^+$ and $\forall x,y \in \mathbb{R}$.

(H6) The multivalued map $F: \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is L^1- Caratheodory and has non-empty compact and convex values such that, there exists a continuous function $h \in C(\mathbb{R}^+,\mathbb{R})$ with $\left\|F\left(t,x\big(\varphi_1(t)\big)\right)\right\| \le h(t)$ a. e. $t \in \mathbb{R}^+$ and $\forall x,y \in \mathbb{R}$.

(H7) There exists a positive real number r_2 such that,

$$r_2 > \frac{\|g_0\| \frac{\|h\|_{L^1}}{2}}{1 - \|\varphi\| \frac{\|h\|_{L^1}}{2}} \qquad (3)$$

where $g_0 = sup_{t\in\mathbb{R}^+}|g(t,0)|$

(H$_8$) The function $v: \mathbb{R}^+ \to \mathbb{R}$ defined by formula $v(t) = \int_0^t (t-s)h(s)\, ds$ is bounded on $\mathbb{R}^+$ and vanish at infinity, i.e. $\lim_{t\to\infty} v(t) = 0$.

Theorem 5.1.1: Assume that the hypothesis (H$_5$) -(H$_8$) hold then second order quadratic functional nonlinear integral inclusion has a solution on $\mathbb{R}^+$provided that,

$$\|\varphi\| \frac{\|h\|_{L^1}}{2} < 1/2 \qquad (4).$$

Moreover, solution second order quadratic functional nonlinear integral inclusion are locally attractive on $\mathbb{R}^+$.

Proof: Let $X = C(\mathbb{R}^+, \mathbb{R})$, transform the problem (2) into a fixed-point problem. Define two operators $\mathcal{A}: X \longrightarrow X$ and $\mathcal{B}: X \longrightarrow \mathcal{P}_{cp,cv}(X)$ by,

$$\mathcal{A}x = \left\{u \in X : u(t) = g\left(t, x(\varphi_2(t))\right)\right\} \qquad (5)$$

$$\mathcal{B}x = \left\{u \in X : u(t) = \int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds \; t \in \mathbb{R}^+ \; and \; f \in S_{F,x}\right\} \quad (6)$$

Then the second order quadratic functional nonlinear integral inclusion(4.1) is equivalent to operator inclusion $x(t) \in \mathcal{A}x(t)\mathcal{B}x(t)$, $t \in \mathbb{R}^+$.

We will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfies all the assumptions of theorem (3.3).

Step I:$\mathcal{A}$ is Lipschitz with Lipschitz constant $K = \|\varphi\|$.

Let $x, y \in X$ then by (H$_5$) we have,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|g\left(t, x(\varphi_2(t))\right) - g\left(t, y(\varphi_2(t))\right)\right|$$

$$\le \varphi(t)\left|x(\varphi_2(t)) - y(\varphi_2(t))\right|$$

For all $t \in \mathbb{R}^+$ taking supremum we have,

$\|\mathcal{A}x - \mathcal{A}y\| \le \|\varphi\|\|x - y\|$ for all $x, y \in \mathbb{R}$

$\mathcal{A}$ is Lipschitz with Lipschitz constant $\|\varphi\|$.

Step II: The multivalued operator $\mathcal{B}$ is compact and upper semi-continuous on X.

To prove $\mathcal{B}$ to be compact using Arzela Ascoli theorem, we must show that $\mathcal{B}$ is uniformly bounded andequicontinuous.

Claim I:$\mathcal{B}$ maps bounded sets into bounded sets in X.

let S be a bounded set in X then $\exists\, r > 0$ such that $\|x\| \le r$ for all $x \in S$.

For each $u \in \mathcal{B}x$ there exists a $f \in S_{F,x}$ such that,

$$u(t) = \int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds$$

For each $t \in \mathbb{R}^+$ by hypothesis (H$_6$) we have

$$|\mathcal{B}x| = |u(t)| = \left|\int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds\right|$$

$$\le \left|\int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds\right|$$

$$\le \frac{\|h\|_{L^1}}{2}$$

This implies that,

$$\|u\| \leq \frac{\|h\|_{L^1}}{2}$$

So $\mathcal{B}x$ is uniformly bounded.

Claim II:$\mathcal{B}$ maps bounded sets into equicontinuous sets.

Let S be a bounded set and $u \in \mathcal{B}x$ for some $x \in S$ then there exists a $f \in S_{F,x}$ such that,

$$u(t) = \int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds$$

Then for any $t_1, t_2 \in \mathbb{R}^+$ we have

$$|u(t_1) - u(t_2)| = \left| \int_0^{t_1} (t_1 - s) f\left(s, x(\varphi_1(s))\right) ds - \int_0^{t_2} (t_2 - s) f\left(s, x(\varphi_1(s))\right) ds \right|$$

$$|u(t_1) - u(t_2)| \leq \frac{\|h\|_{L^1}}{2} |(t_1)^2 - (t_2)^2|$$

The right-hand side of the above inequality tends to zero as $t_1 \longrightarrow t_2$ independent of x therefore by Arzela-Ascoli theorem $\mathcal{B}$ is compact.

Claim III:$\mathcal{B}$ has a closed graph.

Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x_*$. Let $\{u_n\}$ be a sequence such that $u_n \in \mathcal{B}(x_n)$ and $u_n \longrightarrow u_*$ we shall prove that $u_* \in \mathcal{B}(x_*)$. Now since $u_n \in \mathcal{B}(x_n)$ then there exists a $f_n \in S_{F,x_n}$ Such that,

$$u_n(t) = \int_0^t (t-s) f_n\left(s, x_n(\varphi_1(s))\right) ds \quad , \qquad t \in \mathbb{R}^+$$

We must prove that there is a $f_* \in S_{F,x_*}$ such that,

$$u_*(t) = \int_0^t (t-s) f_*\left(s, x_*(\varphi_1(s))\right) ds \quad , \qquad t \in \mathbb{R}^+$$

Consider a continuous linear operator $\Theta: L^1(\mathbb{R}^+, \mathbb{R}) \longrightarrow C(\mathbb{R}^+, \mathbb{R})$ defined by,

$\Theta f(t) = \int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds, \quad t \in \mathbb{R}^+$

Now,

$$\|u_n(t) - u_*(t)\| = \left\| \int_0^t (t-s) f_n\left(s, x_n(\varphi_1(s))\right) ds - \left(\int_0^t (t-s) f_*\left(s, x_*(\varphi_1(s))\right) ds \right) \right\|$$

$\left\| \int_0^t \left(f_n\left(s, x_n(\varphi_1(s))\right) - f_*\left(s, x_*(\varphi_1(s))\right) \right) (t-s) ds \right\| \longrightarrow 0 \quad$ as $n \longrightarrow \infty$

From Lemma it follows that $\Theta \circ S_{F,x}$ is a closed graph operator. Also, from the definition of Θ we have

$$u_* \in \Theta \circ S_{F,x}$$

Since $x_n \longrightarrow x_*$, therefore we have

$u_*(t) = \int_0^t (t-s) f_*\left(s, x_*(\varphi_1(s))\right) ds \quad , \qquad t \in \mathbb{R}^+$

Therefore $\mathcal{B}$ is an upper semi-continuous operator on X.

Thus $\mathcal{B}$ defines an upper semicontinuous and compact and hence is completely continuous multi-valued operator on X. i. e. $\mathcal{B}: X \longrightarrow \mathcal{P}_{cp}(X)$.

Now, we will show that the operator $\mathcal{B}$ has convex values. let $u_1, u_2 \in \mathcal{B}x$ then $\exists\, f_1, f_2 \in S_{F,x}$ such that,

$$u_i(t) = \int_0^t (t-s) f_i\left(s, x(\varphi_1(s))\right) ds \text{ for } i = 1,2$$

Now for any $\lambda \in [0,1]$ we have,

$$\lambda u_1(t) + (1-\lambda)u_2(t) =$$

$$\int_0^t (t-s)\lambda f_1\left(s, x(\varphi_1(s))\right) ds + \int_0^t (t-s)(1-\lambda) f_2\left(s, x(\varphi_1(s))\right) ds$$

$$\lambda u_1(t) + (1-\lambda)u_2(t) = \int_0^t (t-s)\left(\lambda f_1\left(s, x(\varphi_1(s))\right) + (1-\lambda) f_2\left(s, x(\varphi_1(s))\right)\right) ds$$

Since, $F\left(t, x(\varphi_1(t))\right)$ is convex.

$\left(\lambda f_1\left(s, x(\varphi_1(s))\right) + (1-\lambda) f_2\left(s, x(\varphi_1(s))\right)\right) \in F\left(t, x(\varphi_1(t))\right)$ for all $t \in \mathbb{R}^+$

and so that $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{B}x$ which proves the convexity of $\mathcal{B}x$.

Therefore, $\mathcal{B}$ defines a multivalued operator $\mathcal{B}: X \longrightarrow \mathcal{P}_{cv}(X)$.

Therefore, $\mathcal{B}: X \longrightarrow \mathcal{P}_{cp,cv}(X)$

Step III: we will show that $2MK < 1$

By (H_7). Since we have

$$M = \|\mathcal{B}(X)\| = sup\{|\mathcal{B}x|; x \in X\} \le \frac{\|h\|_{L^1}}{2} \quad \text{and} \quad K = \|\varphi\|$$

$$\therefore 2MK \le 2\left[\frac{\|h\|_{L^1}}{2}\|\varphi\|\right]$$

$$< 2 \times \frac{1}{2}$$

$$< 1$$

Thus, all the conditions of theorem (3.3) are satisfied hence conclusion I or conclusion II holds. We will show that conclusion II is not possible.

$$\varepsilon = \{u \in X, \mu u \in \mathcal{A}u\mathcal{B}u, \mu > 1\}$$

Let $u \in \varepsilon$ be arbitrary then we have for $\mu > 1$. $\mu u \in \mathcal{A}u(t)\mathcal{B}u(t)$

Then there exists a $f \in S_{F,x}$ such that for any $\mu > 1$

$u(t) = \mu^{-1}\left[g\left(t, u(\varphi_2(t))\right)\left(\int_0^t (t-s) f\left(s, u(\varphi_1(s))\right) ds\right)\right],\quad$ for $t \in \mathbb{R}^+$

$$|u(t)| \le \left|\frac{1}{\mu}\right| \left|g\left(t, u(\varphi_2(t))\right)\right| \left|\left(\int_0^t (t-s) f\left(s, u(\varphi_1(s))\right) ds\right)\right|$$

$$\frac{1}{\mu} < 1$$

$$|u(t)| \le \left|g\left(t, u(\varphi_2(t))\right) - g(t,0) + g(t,0)\right| \left\{\left|\int_0^t (t-s) f\left(s, u(\varphi_1(s))\right) ds\right|\right\}$$

$$|u(t)| \le \left\{\left|g\left(t, u(\varphi_2(t))\right) - g(t,0)\right| + |g(t,0)|\right\} \left\{\left|\int_0^t (t-s) f\left(s, u(\varphi_1(s))\right) ds\right|\right\}$$

$$\|u\| \le [\|\varphi\|\|u\| + \|g_0\|]\left[\frac{\|h\|_{L^1}}{2}\right]$$

Where $g_0 = sup_{t\in\mathbb{R}^+}|g(t,0)|$

$$\|u\| \le \left[\|\varphi\|\frac{\|h\|_{L^1}}{2}\right]\|u\| + \|g_0\|\frac{\|h\|_{L^1}}{2}$$

$$\|u\| - \left[\|\varphi\|\frac{\|h\|_{L^1}}{2}\right]\|u\| \leq \|g_0\|\frac{\|h\|_{L^1}}{2}$$
$$\left[1 - \|\varphi\|\frac{\|h\|_{L^1}}{2}\right]\|u\| \leq \|g_0\|\frac{\|h\|_{L^1}}{2}$$
$$\|u\| \leq \frac{\|g_0\|\frac{\|h\|_{L^1}}{2}}{1 - \|\varphi\|\frac{\|h\|_{L^1}}{2}}$$

With $\|u\| = r_2$

$$r_2 \leq \frac{\|g_0\|\frac{\|h\|_{L^1}}{2}}{1 - \|\varphi\|\frac{\|h\|_{L^1}}{2}}$$

Which implies that the set ε is bounded. Thus, conclusion II of theorem (3.3) does not hold. Therefore, conclusion I holds. Thus, the operator inclusion $x(t) \in \mathcal{A}x(t)\mathcal{B}x(t)$ consequently problem (2) has a solution on $\mathbb{R}^+$.

Step IV: Finally, we show that the locally attractivity of the solution for second order quadratic functional nonlinear integral inclusion (4.1). Let x and y be two solutions of second order quadratic functional nonlinear integral inclusion (4.1) which is defined on $\mathbb{R}^+$.

Then we have,

$$\begin{aligned}
|x(t) - y(t)| &= \left| g\left(t, x(\varphi_2(t))\right)\int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds \right. \\
&\qquad \left. - g\left(t, y(\varphi_2(t))\right)\int_0^t (t-s) f\left(s, y(\varphi_1(s))\right) ds \right| \\
&\leq \left| g\left(t, x(\varphi_2(t))\right)\int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds \right| \\
&\qquad + \left| g\left(t, y(\varphi_2(t))\right)\int_0^t (t-s) f\left(s, y(\varphi_1(s))\right) ds \right| \\
&\leq \left| g\left(t, x(\varphi_2(t))\right)\right| \int_0^t \left|(t-s) f\left(s, x(\varphi_1(s))\right)\right| ds \\
&\qquad + \left| g\left(t, y(\varphi_2(t))\right)\right| \int_0^t \left|(t-s) f\left(s, y(\varphi_1(s))\right)\right| ds \\
&\leq \left| g\left(t, x(\varphi_2(t))\right)\right| \int_0^t h(s) ds + \left| g\left(t, y(\varphi_2(t))\right)\right| \int_0^t h(s)\, ds
\end{aligned}$$

By hypothesis (H_8) we have,

$$\leq \frac{\varphi}{2}v(t) + \frac{\varphi}{2}v(t)$$
$$|x(t) - y(t)| \leq \varphi v(t)$$

Since $\lim_{t\to\infty} v(t) = 0$ for $\in > 0$ there is real number $T > 0$ such that $v(t) < \frac{\in}{\varphi}$ for all $t \geq T$ then above inequality is,

$$|x(t) - y(t)| < \varphi \frac{\in}{\varphi} = \in$$

Hence completes the proof.

5.2 Existence the Extremal solutions of second order quadratic functional nonlinear inclusion (2)

For existence the extremal solution of second order quadratic functional nonlinear integral inclusion (2)

Definition 5.2.1[5]: A function $a \in C(\mathbb{R}^+, \mathbb{R})$ is called **lower solution** of second order quadratic functional nonlinear integral inclusion (2) if,

$$a(t) \leq g\left(t, a\big(\varphi_2(t)\big)\right)\left(\int_0^t (t-s)v(s)ds\right) \tag{2.1}$$

For all $v \in L^1(\mathbb{R}^+, \mathbb{R})$ such that $v(t) \in F\left(t, a\big(\varphi_1(t)\big)\right)$ almost everywhere for $t \in \mathbb{R}$.

Definition 5.2.2[5]: A function $b \in C(\mathbb{R}^+, \mathbb{R})$ is called **upper solution** of second order quadratic functional nonlinear integral inclusion (2) if,

$$b(t) \geq g\left(t, b\big(\varphi_2(t)\big)\right)\left(\int_0^t (t-s)v(s)ds\right) \tag{2.2}$$

For all $v \in L^1(\mathbb{R}^+, \mathbb{R})$ such that $v(t) \in F\left(t, x\big(\varphi_1(t)\big)\right)$ almost everywhere for $t \in \mathbb{R}$.

Definition 5.2.3[5]: A solution x_M of the second order quadratic functional nonlinear integral inclusion (2) is said to be maximal if x is any other solution of second order quadratic functional nonlinear integral inclusion (2) on $\mathbb{R}^+$ then we have $x\big(\varphi(t)\big) \leq x_M\big(\varphi(t)\big)$for all $t \in \mathbb{R}^+$.

Definition 5.2.4 [5]: A solution x_m of thesecond order quadratic functional nonlinear integral inclusion (4.1) is said to be minimal if x is any other solution of second order quadratic functional nonlinear integral inclusion (4.1) on $\mathbb{R}^+$ then we have $x\big(\varphi(t)\big) \geq x_m\big(\varphi(t)\big)$for all $t \in \mathbb{R}^+$.

We consider the following hypothesis.

(B4) The function $g\left(t, x\big(\varphi_2(t)\big)\right)$ and the multivalued function $F\left(t, x\big(\varphi_1(t)\big)\right)$ are strictly monotone increasing in x almost everywhere for t$\in \mathbb{R}^+$.

(B5) The function $g\left(t, x\big(\varphi_2(t)\big)\right)$ is closed and bounded.

(B6) The second order quadratic functional nonlinear integral inclusion (2) has a lower solution a and an upper solution b with $a \leq b$.

Theorem 5.2.1: Assume that the hypothesis H_5-$H_8$and B_4-B_6 holds then second order quadratic functional nonlinear integral inclusion (2) has a minimal and a maximal solution.

Proof: Let $X = C(\mathbb{R}^+, \mathbb{R})$ and consider an ordered interval $[a, b]$in X which is well defined in view of hypothesis (B_6). Define two operators $\mathcal{A}, \mathcal{B}: [a, b] \longrightarrow \mathcal{P}_{cp}(\mathcal{K})$ as in (5) and (6).

We will show that the operator $\mathcal{A}$ and $\mathcal{B}$ are strictly monotone increasing. Let $x, y \in [a, b]$ be such that $x < y$,

$$\mathcal{A}x = g\left(t, x\big(\varphi_2(t)\big)\right)$$

then by (B_4) $$g\left(t, x(\varphi_2(t))\right) \le g\left(t, y(\varphi_2(t))\right)$$
$$g\left(t, y(\varphi_2(t))\right) = \mathcal{A}y$$
$$\mathcal{A}x \le \mathcal{A}y$$
$$\mathcal{B}x = \int_0^t (t-s) f\left(s, x(\varphi_1(s))\right) ds \, , f \in S_{F,x}$$
$$\mathcal{B}x \le \int_0^t (t-s) f\left(s, y(\varphi_1(s))\right) ds \, , f \in S_{F,y}$$
$$\mathcal{B}x \le \mathcal{B}y$$
Hence the operators $\mathcal{A}$ and $\mathcal{B}$ are strictly monotone increasing.
Next, we will show that the operator $\mathcal{A}$ is compact. Using hypothesis (B_5) it is enough to show that the operator $\mathcal{A}$ is bounded. For that let S is a bounded set in $[a, b]$. For $x \in S$, we have
$$\|x\| \le r$$
$$|\mathcal{A}x| = \left|g\left(t, x(\varphi_2(t))\right) - g(t,0) + g(t,0)\right|$$
$$|\mathcal{A}x| \le \left|g\left(t, x(\varphi_2(t))\right) - g(t,0)\right| + |g(t,0)|$$
$$\|\mathcal{A}x\| \le \|\varphi\|\|x\| + \|g_0\|$$
$$\|\mathcal{A}x\| \le \|\varphi\| r + \|g_0\|$$
Hence $\mathcal{A}$ maps bounded set to bounded set.
Now we can show that as in the proof of theorem (2.4.2) that $\mathcal{B}$ is completely continuous.
Let $x \in [a, b]$ be any element then by (B_6) we have,
$$a \le \mathcal{A}a.\mathcal{B}a \le \mathcal{A}x.\mathcal{B}x \le \mathcal{A}b.\mathcal{B}b \le b$$
which shows that $$\mathcal{A}x.\mathcal{B}x \subset [a, b]$$
Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of theorem (5.2.1) which yields that the operator inclusion $x \in \mathcal{A}x.\mathcal{B}x$ and consequently the second order quadratic functional nonlinear integral inclusion (4.1) has maximal and minimal solution. The completes the proof.

2.6.2 Example of second order quadratic functional nonlinear integral inclusion (4.1)

Consider the following second order quadratic functional nonlinear integral inclusion of type (4.1)
$$x(t) \in \frac{cos(3t)}{8}\left(\frac{x(\varphi_2(t))}{1 + x(\varphi_2(t))} + 2\right)\left[I^2 F\left(t, x(\varphi_1(t))\right)\right]$$
$$t \in \mathbb{R}^+$$
Where $F: \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map and we have $\varphi_1(t) = t$ and $\varphi_2(t) = t^2$ is continuous functiongiven by,
$$t \longrightarrow F\left(t, x(\varphi_1(t))\right) = \left[\frac{|t|^5}{11(|t|^3 + 9)}, \frac{|cost|}{2|cost| + 1}\right]$$
To show hypothesis (H_5) is satisfied,
$$\left|g(t, x(t^2)) - g(t, y(t^2))\right| = \left|\frac{cos(3t)}{8}\left(\frac{x(t^2)}{1 + x(t^2)} + 2\right) - \frac{cos(3t)}{8}\left(\frac{y(t^2)}{1 + y(t^2)} + 2\right)\right|$$
$$\left|g(t, x(t^2)) - g(t, y(t^2))\right| \le \frac{cos(3t)}{8}|x(t^2) - y(t^2)|$$

$$\therefore \varphi(t) = \frac{cos(3t)}{8}$$

$$\|\varphi\| = {}^{1}/_{8}$$

To show that hypothesis (H_6) satisfied,

$$\|F(t, x(t))\| = sup\{|y|; y \in F(t, x(t))\} \leq 1 = h(t)$$

$$\|h\|_{L^1} = 1$$

To show that hypothesis (H_7) is satisfied,

$$r_2 > \frac{\|g_0\| \frac{\|h\|_{L^1}}{2}}{1 - \|\varphi\| \frac{\|h\|_{L^1}}{2}}$$

$$\|g_0\| = sup_{t\in\mathbb{R}^+}|g(t, 0)|$$

$$= sup_{t\in\mathbb{R}^+}\left|\frac{cos(3t)}{8}\left(\frac{0}{1+0} + 2\right)\right|$$

$$= sup_{t\in\mathbb{R}^+}\left|\frac{cos(3t)}{4}\right|$$

$$\|g_0\| = \frac{1}{4}$$

$$r_1 \leq \frac{{}^{1}/_{4} \cdot {}^{1}/_{2}}{1 - {}^{1}/_{8} \cdot {}^{1}/_{2}}$$

$$r_1 \leq \frac{{}^{1}/_{8}}{1 - \frac{1}{16}}$$

$$r_1 \leq \frac{{}^{1}/_{8}}{{}^{15}/_{16}} = \frac{1}{8} \times \frac{16}{15} = \frac{2}{15}$$

$$r_1 \leq 0.1333$$

$$\|\varphi\| \frac{\|h\|_{L^1}}{2} = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16} = 0.0625 < {}^{1}/_{2}$$

Since all the conditions of theorem (4.2) are satisfied hence the above problem has a solution on $\mathbb{R}^+$.

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