International Research Journal of Management Science & Technology

ISSN 2250 – 1959(0nline)
2348 – 9367 (Print)

An Internationally Indexed Peer Reviewed & Refereed Journal

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Published by iSaRa Solutions

A Refinement of Trapezoidal Rule

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Abstract: In this paper a new numerical integration method is proposed that provides better estimates as compared to Newton Cotes method of numerical integration. This method is inspired by Trapezoidal Rule where after segmentation, upper part of each segment is again subdivided into rectangles and triangles. The area of each segment is then obtained as sum of areas of these geometric shapes and area of lower part, which is a rectangle. The process resulted in another formula for numerical integration.

AMS classification: 26D15, 26D20.

Key words and phrases: Numerical integration, Trapezoidal rule, Newton – Cotes formula.

1. **Introduction**

Integration is the process of measuring the area under a function plotted on a graph. The process also known as Integral Calculus, has countless applications in a wide range of fields including engineering, statistics, finances, physics etc.

Sometimes, the evaluation of expressions involving these integrals can become very difficult, but not impossible. Due to this reason a number of different numerical integration methods have been developed to simplify the integrals.

Numerical integration involves the approximation of numerical values that can not be integrated analytically [1]. Several numerical integration methods such ads Newton – Cotes, Romberg integration, Gauss Quadrature and Monte Carlo integration uses interpolating polynomials. Newton – Cotes method such as Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule and Bool's rule are special cases of 1st, 2nd, 3rd and 4th order polynomials used respectively. The Trapezoidal rule and numerical integration method proposes here have no restrictions on number of segmentations. The number of segments for Simpson's 1/3 rule must be even and for Simpson's 3/8 rule, the number of segments must be multiple of 3. For Bool's and Weddle's rule the number of segments must be multiple of 4 and 6 respectively.

The Newton – Cotes formula involves $n+1$ points in the interval $[a,b]$ with n order polynomial, which passes through the equally spaced points $x_i\ (i=1,2,\cdots,n)$. That is, if we divide the (a,b) into n sub-intervals of width h so that $x_0=a,\ x_1=x_0+h, x_2=x_0+2h,\cdots,x_n=x_0+nh=b$. Then

$$I=\int_a^b f(x)\,dx = nh\begin{bmatrix} y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+ \\ \left(\frac{n^4}{5}-\frac{3n^3}{2}+\frac{11n^2}{3}-3n\right)\frac{\Delta^4}{4}y_0+\cdots\cdots \end{bmatrix} \quad (1.1)$$

This is known as Newton – Cotes formula [2], [3]. From this formula, we can deduce the following important rules by taking $n = 1,2,3,\cdots$

When $n = 1$, we have a simple Trapezoidal rule

$$I = \int_a^b f(x)\,dx = \frac{h}{2}[(f_0 + f_1) + 2(f_2 + f_3 + \cdots + f_{n-1})]. \tag{1.2}$$

When $n = 2$, we have a Simpson's 1/3 rule

$$I = \int_a^b f(x)\,dx = \frac{h}{3}[(f_0 + f_n) + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2})]. \tag{1.3}$$

When $n = 3$, we have a Simpson's 3/8 rule

$$I = \int_a^b f(x)\,dx = \frac{3h}{8}\begin{bmatrix}(f_0 + f_n) + 3(f_1 + f_2 + f_4 + \cdots + f_{n-1}) + \\ 2(f_3 + f_6 + \cdots + f_{n-3})\end{bmatrix}. \tag{1.4}$$

The main objective of this paper is to propose a numerical integration method that provide better estimates as compared to the Newton – Cotes method of integration.

2. Numerical Integration formula

Let us divide the interval (a, b) into n $(n \in Z^+)$ sub-intervals each of width $h = \frac{b-a}{n}$. Define x_i by $x_i = a + ih,\ i = 1,2,\cdots,n$. Then $x_0 = a$ and $x_n = b$. Let $f_i = f(x_i),\ i = 1,2,\cdots,n$ be the ordinates at $x_i,\ i = 1,2,\cdots,n$ of the function f. Suppose also that the interval $[x_i, x_{i+1}],\ i = 1,2,\cdots,n-1$ is further divided into k equidistant points (sub-intervals) $x_i + \frac{t}{k}h,\ t = 1,2,\cdots,k$. Then the corresponding ordinates of f are given by $f_{i+\frac{t}{k}} = f\left(x_i + \frac{t}{k}h\right),\ t = 1,2,\cdots,k; i = 0,1,\cdots,n-1$. Clearly when $t = k, x_i + \frac{k}{k}h = x_{i+1}$ and $f_{i+\frac{k}{k}} = f_{i+1}$.

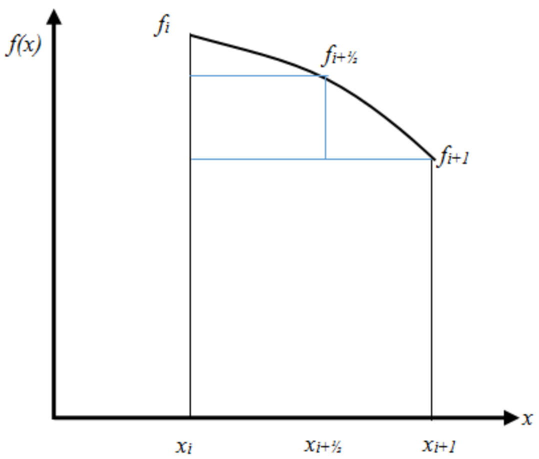

Figure 1(a)

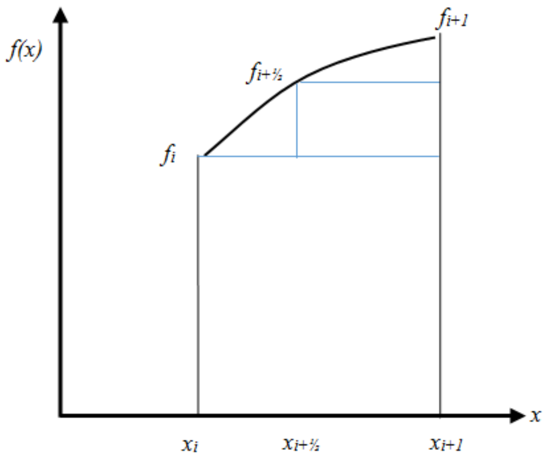

Figure 1(b)

In Figure 1(a), f is monotone deceasing function and in Figure 1(b) f is monotone increasing function.

Now, the area under the curve for ith strip is estimated as

For $k = 1$, the area (A_i) under the ith strip is

When f is monotone deceasing function in (x_i, x_{i+1})

$$A_i = hf_{i+1} + \frac{h}{2}(f_i - f_{i+1}) = \frac{h}{2}(f_i + f_{i+1}).$$

When f is monotone increasing function in (x_i, x_{i+1})

$$A_i = hf_{i+1} + \frac{h}{2}(f_{i+1} - f_i) = \frac{h}{2}(f_i + f_{i+1}).$$

For $k = 2$, the area (A_i) under the ith strip is given as

When f is monotone deceasing function in (x_i, x_{i+1})

$$A_i = hf_{i+1} + \frac{h}{2}\left(f_{i+\frac{1}{2}} - f_{i+1}\right) + \frac{h}{4}\left(f_{i+\frac{1}{2}} - f_{i+1}\right) + \frac{h}{4}\left(f_i - f_{i+\frac{1}{2}}\right)$$
$$= \frac{h}{4}\left(f_i + 2f_{i+\frac{1}{2}} + f_{i+1}\right).$$

When f is monotone increasing function in (x_i, x_{i+1})

$$A_i = hf_i + \frac{h}{2}\left(f_{i+\frac{1}{2}} - f_i\right) + \frac{h}{4}\left(f_{i+\frac{1}{2}} - f_i\right) + \frac{h}{4}\left(f_{i+1} - f_{i+\frac{1}{2}}\right)$$
$$= \frac{h}{4}\left(f_i + 2f_{i+\frac{1}{2}} + f_{i+1}\right).$$

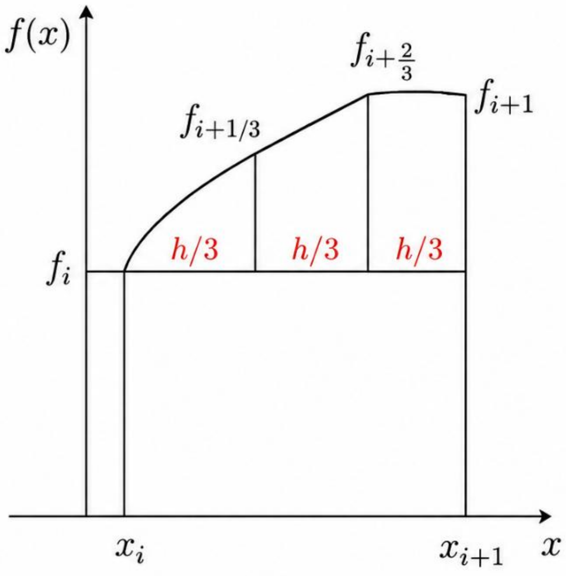

Fig 1(a): f is monotone increasing

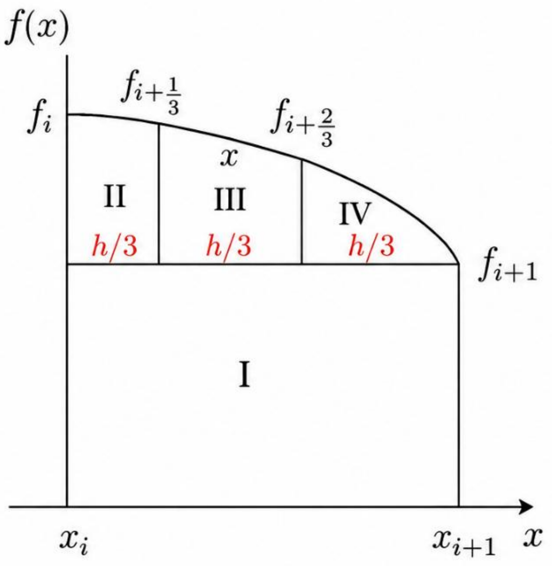

Fig 1(b): f is monotone decreasing

Clearly, the result is same irrespective of whether f is monotone increasing or decreasing.

For $k = 3$, the area (A_i) under the ith strip is given as

When f is monotone deceasing function in (x_i, x_{i+1}), then

$$A_i = hf_{i+1} + \frac{h}{3}\left(f_{i+\frac{1}{3}} - f_{i+1}\right) + \frac{h}{3}\left(f_{i+\frac{2}{3}} - f_{i+1}\right) + \frac{h}{6}\left(f_{i+\frac{2}{3}} - f_{i+1}\right)$$
$$+ \frac{h}{6}\left(f_{i+\frac{1}{3}} - f_{i+\frac{2}{3}}\right) + \frac{h}{6}\left(f_i - f_{i+\frac{1}{3}}\right)$$
$$= \frac{h}{6}\left(f_i + 2\left(f_{i+\frac{1}{3}} f_{i+\frac{2}{3}}\right) + f_{i+1}\right).$$

When f is monotone increasing function in (x_i, x_{i+1}), then

$$A_i = hf_i + \frac{h}{3}\left(f_{i+\frac{2}{3}} - f_i\right) + \frac{h}{3}\left(f_{i+\frac{1}{3}} - f_i\right) + \frac{h}{6}\left(f_{i+\frac{1}{3}} - f_i\right)$$
$$+\frac{h}{6}\left(f_{i+\frac{2}{3}} - f_{i+\frac{1}{3}}\right) + \frac{h}{6}\left(f_{i+1} - f_{i+\frac{2}{3}}\right)$$
$$= \frac{h}{6}\left(f_i + 2\left(f_{i+\frac{1}{3}} + f_{i+\frac{2}{3}}\right) + f_{i+1}\right)$$

Clearly, the result is same irrespective of whether f is monotone increasing or decreasing.
For $k = 3$, the area (A_i) under the ith strip is estimated as

$$A_i = \frac{h}{2k}\left(f_i + 2\sum_{t=1}^{k-1} f_{i+\frac{t}{k}} + f_{i+1}\right),\ i = 1,2,\cdots,n; k = 1,2,\cdots \tag{2.1}$$

3. Composite Numerical Integration Formula

The composite method provides a formula for estimating numerically the area under the curve of f and above the horizontal axis between the interval $[a,b]$. It is the sum of areas of all n strips each of width $h = \frac{b-a}{n}$ and k sub-divisions at the top as indicated in the figures.

$$A_n = \int_a^b f(x)\,dx = \sum_{i=1}^{n-1}\left(\int_{x_i}^{x_{i+1}} f(x)\,dx\right)$$
$$A_n = \frac{h}{2k}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=1}^{n-1}\sum_{t=0}^{k-1} f_{i+\frac{t}{k}} + f_n\right). \tag{3.1}$$

When $k = 2$, (3.1) reduces to

$$A_n = \int_a^b f(x)\,dx = \frac{h}{4}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=0}^{n-1} f_{i+\frac{1}{2}} + f_n\right). \tag{3.2}$$

When $k = 3$, (3.1) reduces to

$$A_n = \int_a^b f(x)\,dx = \frac{h}{6}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=0}^{n-1} f_{i+\frac{1}{3}} + 2\sum_{i=0}^{n-1} f_{i+\frac{2}{3}} + f_n\right). \tag{3.3}$$

And so on.

4. Example

Consider the integral

$$I = \int_0^2 e^{x^2}\,dx$$

The exact value is $I = 16.45263$

If we divide the $[0,2]$ into 12 sub-intervals each of width $h = \frac{b-a}{n} = \frac{2}{12} = 0.1667$, then the estimate of integral using Trapezoidal rule is given by $I = 16.95311$, using Simpson's 1/3 rule $I = 16.47144$, using Simpson's 3/8 rule $I = 16.49168$.

By our method using (3.1), we have for $k = 2, I = 16.57869$ and for $k = 9,10,11,\cdots$, $I = 16.458870, 16.456810, 16.45680, \cdots$.

Therefore, proposed method (3.1), gives better estimate.

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